Introduction to Mathematical Quantum Theory

Solution to the Exercises

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Exercise 1

a Consider the function $f \in L^1(\mathbb{T})$ defined as the periodization of

$$f(x) := x(2\pi - x). \tag{1}$$

Calculate the Fourier coefficients of f and use them to prove that

$$\sum_{k=0}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{2}$$

b Let σ be a positive real number and $\mathbf{v},\ \mathbf{u} \in \mathbb{R}^d$. Consider the function $g_{\sigma,\mathbf{v},\mathbf{u}}$ in the space $L^2\left(\mathbb{R}^d\right)$ with $d \in \mathbb{N}$ defined as

$$g_{\sigma,\mathbf{v},\mathbf{u}}(\mathbf{x}) := \left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}-\mathbf{v}|^2 + i\mathbf{u}\cdot\mathbf{x}}. \tag{3}$$

Then prove that $\widehat{g}_{\sigma,\mathbf{v},\mathbf{u}} = e^{i\mathbf{v}\cdot\mathbf{u}} g_{\sigma^{-1},\mathbf{u},-\mathbf{v}}$, i.e.

$$\mathcal{F}\left[\left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}-\mathbf{v}|^2 + i\mathbf{u}\cdot\mathbf{x}}\right](\mathbf{k}) = \left(\frac{1}{\sigma\pi}\right)^{\frac{d}{4}} e^{-\frac{1}{2\sigma}|\mathbf{k}-\mathbf{u}|^2 - i\mathbf{u}\cdot(\mathbf{k}-\mathbf{v})}. \tag{4}$$

Proof. For the proof of **a**, first consider the coefficients of f; if $k \in \mathbb{Z}\backslash\{0\}$ those are given as

$$\begin{aligned}
\widehat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x(2\pi - x) e^{-ikx} dx \\
&= \frac{i}{\sqrt{2\pi}k} \left[x(2\pi - x) e^{-ikx}\right]_0^{2\pi} - \frac{\sqrt{2}i}{\sqrt{\pi}k} \int_0^{2\pi} (\pi - x) e^{-ikx} dx \\
&= \frac{\sqrt{2}}{\sqrt{\pi}k^2} \left[(\pi - x) e^{-ikx}\right]_0^{2\pi} + \frac{\sqrt{2}}{\sqrt{\pi}k^2} \int_0^{2\pi} e^{-ikx} dx \\
&= \frac{\sqrt{2}}{\sqrt{\pi}k^2} \left[-\pi e^{-2\pi ki} - \pi + \frac{i}{k}\left(e^{-2\pi ki} - 1\right)\right] = -\frac{\sqrt{8\pi}}{k^2}.
\end{aligned}$$

On the other hand when $k = 0$ we have

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x(2\pi - x) dx = \frac{1}{\sqrt{2\pi}} \left[\pi x^2 - \frac{1}{3}x^3\right]_0^{2\pi} = \frac{\sqrt{8\pi}\pi^2}{3}.$$

We then use the fact that $f(0) = 0$ to get

$$0 = f(0) = \sum_{k\in\mathbb{Z}} \widehat{f}(k) = \frac{\sqrt{8\pi}\pi^2}{3} - 2\sqrt{8\pi}\sum_{k=0}^{+\infty}\frac{1}{k^2}$$
$$\implies \sum_{k=0}^{+\infty}\frac{1}{k^2} = \frac{\pi^2}{6},$$

which concludes the proof of (2).

For the proof of **b**, recall that for any positive real number $\alpha > 0$ we have

$$\int_{\mathbb{R}^d} e^{-\alpha|\mathbf{x}|^2} d\mathbf{x} = \left(\frac{\pi}{\alpha}\right)^{\frac{d}{2}}.$$

Consider now the function

$$h_\sigma(\mathbf{x}) := g_{\sigma,\mathbf{0},\mathbf{0}}(\mathbf{x}) = \left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}|^2}.$$

In general we have that

$$\partial_{x_j} h_\sigma(\mathbf{x}) = -\sigma x_j h_\sigma(\mathbf{x}).$$

Consider then the derivative on the j-th component of $\widehat{h}_\sigma$. Now, given that h_σ is an exponentially decaing continuous function, we can apply Leibniz theorem and integration by part to get

$$\begin{aligned}
\partial_{k_j}\widehat{h}_\sigma(\mathbf{k}) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} h_\sigma(\mathbf{x})\,\partial_{k_j}\left(e^{-i\mathbf{k}\cdot\mathbf{x}}\right)d\mathbf{x}\\
&= -i\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} x_j h_\sigma(\mathbf{x})\,e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\\
&= \frac{i}{\sigma}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \partial_{x_j} h_\sigma(\mathbf{x})\,e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\\
&= -\frac{i}{\sigma}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} h_\sigma(\mathbf{x})\,\partial_{x_j} e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\\
&= -\frac{1}{\sigma}k_j\widehat{h}_\sigma(\mathbf{k}).
\end{aligned}$$

This is a well defined differential equation, with initial datum

$$\widehat{h}_\sigma(\mathbf{0}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} h_\sigma(\mathbf{x})\,d\mathbf{x} = \left(\frac{1}{\sigma\pi}\right)^{\frac{d}{4}}$$

If we now suppose that $\widehat{h}_\sigma(\mathbf{k}) = f_1(k_1)\cdot\ldots\cdot f_d(k_d)$, we get that for any j

$$f_j'(t) = -\frac{1}{\sigma}t f_j(t),$$

and therefore, integrating t between 0 and k_j we get

$$-\frac{k_j^2}{2\sigma} = -\int_0^{k_j} \frac{1}{\sigma} t dt = \int_0^{k_j} \frac{f_j'(t)}{f_j(t)} dt = [\log(f_j(t))]_0^{k_j} = \log\left(\frac{f_j(k_j)}{f_j(0)}\right),$$

and therefore we get

$$\begin{aligned} & f_j(k_j) = f_j(0)\, e^{-\frac{k_j^2}{2\sigma}} \\ \Rightarrow & \widehat{h}_\sigma(\mathbf{k}) = \prod_{j=1}^d \left(f_j(0)\, e^{-\frac{k_j^2}{2\sigma}}\right) = \widehat{h}_\sigma(\mathbf{0})\, e^{-\frac{|\mathbf{k}|^2}{2\sigma}} = \left(\frac{1}{\sigma\pi}\right)^{\frac{d}{4}} e^{-\frac{|\mathbf{k}|^2}{2\sigma}} = h_{\sigma^{-1}}(\mathbf{k}). \end{aligned}$$

Recall now that for any vector $\mathbf{r} \in \mathbb{R}^d$ the operators $T_\mathbf{r}$ and $M_\mathbf{r}$ are defined as

$$T_\mathbf{r} f(\mathbf{x}) := f(\mathbf{x} - \mathbf{r}), \qquad M_\mathbf{r} f(\mathbf{x}) := e^{-i\mathbf{r}\cdot\mathbf{x}} f(\mathbf{x}), \qquad \forall f \in L^1\left(\mathbb{R}^d\right).$$

Then, we saw before that

$$\mathcal{F} T_\mathbf{r} = M_\mathbf{r} \mathcal{F}, \qquad \mathcal{F} M_\mathbf{r} = T_{-\mathbf{r}} \mathcal{F}.$$

We now get to calculate the transform of $g_{\sigma,\mathbf{v},\mathbf{u}}$. First notice that $g_{\sigma,\mathbf{v},\mathbf{u}} = M_{-\mathbf{u}} T_\mathbf{v} g_{\sigma,\mathbf{0},\mathbf{0}} = M_{-\mathbf{u}} T_\mathbf{v} h_\sigma$. Notice now that for any $f \in L^2\left(\mathbb{R}^d\right)$ we get

$$(T_\mathbf{u} M_\mathbf{v} f)(\mathbf{x}) = (M_\mathbf{v} f)(\mathbf{x} - \mathbf{u}) = e^{-i\mathbf{v}\cdot(\mathbf{x}-\mathbf{u})} f(\mathbf{x} - \mathbf{u}) = e^{i\mathbf{v}\cdot\mathbf{u}} (M_\mathbf{v} T_\mathbf{u} f)(\mathbf{x}).$$

We then have

$$\widehat{g}_{\sigma,\mathbf{v},\mathbf{u}} = \mathcal{F} M_{-\mathbf{u}} T_\mathbf{v} h_\sigma = T_\mathbf{u} M_\mathbf{v} h_{\sigma^{-1}} = e^{i\mathbf{v}\cdot\mathbf{u}} M_\mathbf{v} T_\mathbf{u} h_{\sigma^{-1}} = e^{i\mathbf{v}\cdot\mathbf{u}} g_{\sigma^{-1},\mathbf{u},-\mathbf{v}},$$

which concludes the proof.

□

Exercise 2

Consider V_1 and V_2 two normed vector spaces over[1] $\mathbb{F}$ and $T : V_1 \to V_2$ a linear mapping. Define $\|T\|_{V_1,V_2}$ as

$$\|T\| := \sup_{v \in V_1,\ v \neq 0} \frac{\|Tv\|}{\|v\|}. \tag{5}$$

For a generic linear mapping T we have $\|T\| \in [0, +\infty]$. Prove that

$$\begin{aligned} \|T\| &= \sup_{v \in V_1,\ \|v\|_{V_1} = 1} \|Tv\| & (6) \\ &= \sup_{v \in V_1,\ \|v\|_{V_1} \leqslant 1} \|Tv\|. & (7) \end{aligned}$$

Prove moreover that the following are equivalent

[1]Here and in the following $\mathbb{F}$ can be chosen to be either $\mathbb{R}$ or $\mathbb{C}$.

a T is continuous.

b T is continuous in 0, meaning that for any sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq V_1$,

$$v_n \to 0 \quad \Longrightarrow \quad Tx_n \to 0. \tag{8}$$

c The quantity $\|T\|$ is finite, meaning that $\|T\| < +\infty$.

Proof. To prove (6) we get

$$\begin{aligned}\|T\| &= \sup_{v\in V_1,\ v\neq 0} \frac{\|Tv\|}{\|v\|} = \sup_{v\in V_1,\ v\neq 0} \left\| T\left(\frac{v}{\|v\|}\right)\right\| \\ &= \sup_{v\in V_1,\ \|v\|=1} \|Tv\| .\end{aligned}$$

To prove (7) first notice that given that the unit sphere is a subset of the corresponding unit ball we have

$$\sup_{v\in V_1,\ \|v\|=1} \|Tv\| \leqslant \sup_{v\in V_1,\ \|v\|\leqslant 1} \|Tv\| .$$

On the other hand, suppose that $v \in V_1$ with $\|v\| \leqslant 1$, then

$$\sup_{v\in V_1,\ \|v\|\leqslant 1} \|Tv\| \leqslant \sup_{v\in V_1,\ \|v\|\leqslant 1} \|T\| \, \|v\| = \|T\| ,$$

which concludes the proof of the first part of the exercise.

Next notice that **a** implies **b** trivially.

To prove that **b** implies **c**, we have that if T is continuous, the preimage of any open set is open. In particular, consider[2] $T^{-1}(B_1(0))$. Given that $0 \in T^{-1}(B_1(0))$ and T is continuous, there exists a positive real number R such that $B_R(0) \subseteq T^{-1}(B_1(0))$, or equivalently, by linearity of T, that $T(B_1(0)) \subseteq B_{R^{-1}}(0)$. This can be also written as

$$\|v\| \leqslant 1 \Longrightarrow \|Tv\| \leqslant \frac{1}{R}$$

and implies in particular that

$$\|T\| = \sup_{v\in V_1,\ \|v\|\leqslant 1} \|Tv\| \leqslant \frac{1}{R},$$

which implies **c**.

To prove that **c** implies **a**, consider a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq V_1$ such that $v_n \to v$ in V_1. Then we have

$$\|Tv_n - Tv\| = \|T(v_n - v)\| \leqslant \|T\| \, \|v_n - v\| \to 0,$$

completing the proof of the exercise.

□

[2]Recall that $B_r(v)$ denote the ball of radius r around v.

Exercise 3 (Young Inequality)

Consider $p,\ q,\ r \in [1, +\infty]$ such that

$$\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}. \tag{9}$$

Let $f \in L^q\left(\mathbb{R}^d\right)$, $g \in L^r\left(\mathbb{R}^d\right)$; prove that

$$\|f * g\|_p \leqslant \|f\|_q \|g\|_r. \tag{10}$$

Hint: Consider the functions $\alpha,\ \beta,\ \gamma$ *defined as*

$$\alpha(\mathbf{x}, \mathbf{y}) := |f(\mathbf{y})|^q |g(\mathbf{x} - \mathbf{y})|^r, \tag{11}$$

$$\beta(\mathbf{y}) := |f(\mathbf{y})|^q, \tag{12}$$

$$\gamma(\mathbf{x}, \mathbf{y}) := |g(\mathbf{x} - \mathbf{y})|^r, \tag{13}$$

notice that

$$|f * g(\mathbf{x})| \leqslant \int_{\mathbb{R}^d} \alpha(\mathbf{x}, \mathbf{y})^{\frac{1}{p}} \beta(\mathbf{y})^{\frac{p-q}{pq}} \gamma(\mathbf{x}, \mathbf{y})^{\frac{p-r}{pr}} \, d\mathbf{y} \tag{14}$$

and that

$$\frac{1}{p} + \frac{p-q}{pq} + \frac{p-r}{pr} = 1 \tag{15}$$

to apply Hölder inequality.

Proof. Consider α, β and γ as in the Hint. From basic algebraic properties of the Hölder conjugate exponents we get that

$$\alpha(\mathbf{x}, \mathbf{y}) \beta(\mathbf{y}) \gamma(\mathbf{x}, \mathbf{y}) = |f(\mathbf{y}) g(\mathbf{x} - \mathbf{y})|.$$

Given that

$$\frac{1}{p} + \frac{p-q}{pq} + \frac{p-r}{pr} = \frac{1}{q} + \frac{1}{r} - \frac{1}{p} = 1,$$

applying the previous equality to (10) and using Hölder inequality we get

$$\begin{aligned}
|f * g(\mathbf{x})| &\leqslant \int_{\mathbb{R}^d} \alpha(\mathbf{x}, \mathbf{y})^{\frac{1}{p}} \beta(\mathbf{y})^{\frac{p-q}{pq}} \gamma(\mathbf{x}, \mathbf{y})^{\frac{p-r}{pr}} \, d\mathbf{y} \\
&\leqslant \left\|\alpha(\mathbf{x}, \cdot)^{\frac{1}{p}}\right\|_p \left\|\beta^{\frac{p-q}{pq}}\right\|_{\frac{pq}{p-q}} \left\|\gamma(\mathbf{x}, \cdot)^{\frac{p-r}{pr}}\right\|_{\frac{pr}{p-r}} \\
&= \|\alpha(\mathbf{x}, \cdot)\|_1^{\frac{1}{p}} \|f\|_q^{\frac{p-q}{p}} \|g\|_r^{\frac{p-r}{p}}.
\end{aligned}$$

Now expanding the norm of α we get that

$$\begin{aligned}
\|\alpha\|_1 &= \int_{\mathbb{R}^{2d}} |f(\mathbf{y})|^q |g(\mathbf{x} - \mathbf{y})|^r \, d\mathbf{x} d\mathbf{y} \\
&= \|f\|_q^q \|g\|_r^r.
\end{aligned}$$

So now we get

$$\|f * g\|_p = \left[\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} f(\mathbf{y})\, g(\mathbf{x}-\mathbf{y})\, d\mathbf{x}\right]^p d\mathbf{y}\right]^{\frac{1}{p}} \leqslant \|f\|_q^{\frac{p-q}{p}} \|g\|_r^{\frac{p-r}{p}} \left[\int_{\mathbb{R}^d} \|\alpha(\mathbf{x},\cdot)\|_1\, d\mathbf{y}\right]^{\frac{1}{p}}$$
$$= \|f\|_q^{\frac{p-q}{p}} \|g\|_r^{\frac{p-r}{p}} \|g\|_r^{\frac{r}{p}} \|f\|^{\frac{q}{p}} = \|f\|_q \|g\|_r ,$$

which concludes our proof.

□

Exercise 4

a Prove that there exists a positive real number C such that we have

$$\sup_{0\leqslant a<b<+\infty}\left|\int_a^b \frac{\sin x}{x}dx\right| \leqslant C. \tag{16}$$

Hint: Consider the function

$$F(t) := \int_0^\eta e^{-tx}\frac{\sin x}{x}dx. \tag{17}$$

Deduce a bound on $F'(t)$ uniform in η. Apply the fundamental theorem of calculus for $F(0)$ to conclude.

b Consider an odd function $f \in L^1(\mathbb{R})$. Prove that for any such function we have

$$\sup_{0\leqslant a<b<+\infty}\left|\int_a^b \frac{\widehat{f}(k)}{k}dk\right| \leqslant \frac{C}{(2\pi)^{\frac{d}{2}}}\|f\|_1 . \tag{18}$$

c Let $g(k)$ be a continuous odd function on the line such that is equal to $1/\log k$ for any $k \geqslant 2$. Prove that there cannot be an $L^1(\mathbb{R})$ function whose Fourier transform is g.

Proof. We first prove **a**; given that the function sinc is even, it is enough to bound the following quantity:

$$\left|\int_0^\eta \frac{\sin x}{x}dx\right|,$$

with η a positive real number.

Consider now the function $F(t)$ defined as

$$F(t) := \int_0^\eta e^{-tx}\frac{\sin x}{x}dx.$$

Then $F(t)$ is well defined and continuous for any real number t and we have that $F(0)$ is our initial quantity and $F(t) \to 0$ as $t \to +\infty$. Moreover the derivative of F gives

$$F'(t) = \int_0^\eta e^{-tx}\sin x\; dx = -\mathrm{Im}\left(\int_0^\eta e^{-(t+i)x}dx\right)$$
$$= \frac{1}{1+t^2}\left(1 - te^{-\eta t}\sin\eta - e^{-\eta t}\cos\eta\right).$$

Using now the fundamental theorem of calculus we get

$$F(0) = F(T) + \int_T^0 F'(t)\, dt$$

for any positive T and hence, taking the limit $T \to +\infty$

$$\begin{aligned} F(0) &= \lim_{T\to+\infty} F(0) \\ &= \lim_{T\to+\infty} \left[F(T) - \int_0^T F'(t)\, dt \right] \\ &= -\int_0^{+\infty} F'(t)\, dt \\ &= \int_0^{+\infty} \frac{1}{1+t^2} \left(te^{-\eta t} \sin\eta + e^{-\eta t}\cos\eta - 1 \right) dt. \end{aligned}$$

For any positive real number η we get that

$$\begin{aligned} \sup_{t>0} \left| te^{-\eta t} \sin\eta \right| &= \left| \frac{\sin\eta}{\eta} \right| \sup_{t>0}\ te^{-t} = \left| \frac{\sin\eta}{\eta e} \right| \leqslant e^{-1} \\ \sup_{t>0} \left| e^{-\eta t}\cos\eta - 1 \right| &= \sup_{t>0} \left(1 - e^{-t}\cos\eta \right) = 1 \end{aligned}$$

and therefore we can bound $|F(0)|$ as

$$\begin{aligned} \left| \int_0^\eta e^{-tx} \frac{\sin x}{x} dx \right| = |F(0)| &\leqslant \frac{1+e}{e} \int_0^{+\infty} \frac{1}{1+t^2} dt \\ &= \frac{\pi(1+e)}{2e}. \end{aligned}$$

Next, to prove **b** consider f an odd function. Then we have

$$f(x) = -f(-x) \quad \Longrightarrow \quad f(x) = \frac{1}{2}\left(f(x) - f(-x) \right).$$

This implies that if we consider the Fourier transform of f we get

$$\begin{aligned} \widehat{f}(k) &= \frac{1}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}} \left[f(x) - f(-x) \right] e^{-ikx} dx \\ &= \frac{1}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}} f(x) \left[e^{-ikx} - e^{ikx} \right] dx \\ &= \frac{i}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}} f(x) \sin(kx)\, dx \\ &= \frac{2i}{(2\pi)^{\frac{d}{2}}} \int_0^{+\infty} f(x) \sin(kx)\, dx. \end{aligned}$$

We substitute this in (18) to get

$$\begin{aligned}\left|\int_a^b \frac{\widehat{f}(k)}{k}dk\right| &= \frac{2}{(2\pi)^{\frac{d}{2}}}\left|\int_a^b\int_0^{+\infty} f(x)\,\frac{\sin(kx)}{k}dxdk\right| \\ &\leqslant \frac{2}{(2\pi)^{\frac{d}{2}}}\int_0^{+\infty}|f(x)|\int_a^b\left|\frac{\sin(kx)}{k}\right|dkdx \\ &= \frac{2}{(2\pi)^{\frac{d}{2}}}\int_0^{+\infty}|f(x)|\int_{xa}^{xb}\left|\frac{\sin\ k}{k}\right|dkdx \\ &\leqslant \frac{2\Xi}{(2\pi)^{\frac{d}{2}}}\int_0^{+\infty}|f(x)|\,dx = \frac{\Xi}{(2\pi)^{\frac{d}{2}}}\,\|f\|_1\end{aligned}$$

where in the last inequality we used (16). This concludes the proof of **b**.

To prove **c** now, suppose $g = \widehat{h}$. Then on one hand from (18) for any positive real number $R > 2$ we would have

$$\left|\int_2^R \frac{g(k)}{k}dk\right| = \left|\int_2^R \frac{\widehat{h}(k)}{k}dk\right| \leqslant \frac{\Xi}{(2\pi)^{\frac{d}{2}}}\,\|h\|_1\,.$$

On the other hand, we get that

$$\left|\int_2^R \frac{g(k)}{k}dk\right| = \int_2^R \frac{1}{k\log k}dk = \int_{\log 2}^{\log R}\frac{1}{z}dz = \log\frac{\log R}{\log 2},$$

where in the second equality we used the change of variables $z = \log k$. Now the last term goes to infinity as R goes to infinity, but this is absurd given that we proved above that it should be bounded uniformly in R. Therefore such an h does not exists and the proof is complete.

□